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Three positive solutions of semilinear elliptic equations in the half space with a hole

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Abstract

In this paper, assume that h is nonnegative and $\|h\|_{L^2} > 0$, we prove that if $\|h\|_{L^2}$ is sufficiently small, then there are at least three positive solutions of Eq. (1) in $\mathbb{R}^N_+ \backslash \overline{D}$, where D is a $C^{1,1}$ bounded domain in $\mathbb{R}^N_+$.

Keywords: Three positive solutions; Semilinear elliptic equations; Minimax method; (PS)-sequence

1. Introduction

Let $\mathbb{R}^N_+ = \{(z', z_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid z_N > 0\}$ be the upper half space. Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + h(z) & \text{in } \Omega; \\ u \in H_0^1(\Omega), \end{cases} \tag{1}$$

where $\Omega = \mathbb{R}^N_+ \backslash \overline{D}$, D is a $C^{1,1}$ bounded domain in $\mathbb{R}^N_+$, $2 < p < 2^* = 2N/(N-2)$ for $N \geqslant 3$. Let

$$d(p, \alpha) = (p-2)\left(\frac{1}{p-1}\right)^{\frac{p-1}{p-2}} \left(\frac{2p}{p-2}\right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}}, \tag{2}$$

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$h(z) \geqslant 0$ and $0 < \|h\|_{L^2} < d(p, \alpha)$. Associated with Eq. (1), we consider the functionals a, b and J_h, for $u \in H_0^1(\Omega)$,

$$a(u) = \int_\Omega \left(|\nabla u|^2 + u^2\right); \qquad b(u) = \int_\Omega |u|^p; \qquad J_h(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u_+) - \int_\Omega hu.$$

By Rabinowitz [11, Proposition B.10.], a, b and J_h are of C^2. For $h = 0$, we consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{3}$$

and the energy functional $J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u_+)$. Esteban and Lions [6] proved that there is not any nontrivial solution of Eq. (3) in $\mathbb{R}_+^N$ (Esteban–Lions domain). Wang [14] proved that if ρ is sufficiently small and $z_{0N} \to \infty$, then Eq. (3) admits a positive higher energy solution in $\mathbb{R}_+^N \backslash \overline{B_\rho(z_0', z_{0N})}$, where $(z_0', z_{0N}) = (z_{01}, z_{02}, \ldots, z_{0N})$, $B_\rho(z_0', z_{0N}) = \{z \in \mathbb{R}^N: |z - (z_0', z_{0N})| < \rho\}$.

For $h \neq 0$, suppose that h is nonnegative, small and exponential decay, Zhu [17] and Hsu and Wang [7] proved that Eq. (1) admits at least two positive solutions in $\mathbb{R}^N$, an exterior strip domain $\mathbf{A}^r \backslash \overline{D}$, respectively. Without the condition of exponential decay, Cao and Zhou [5] proved that Eq. (1) admits at least two positive solutions in $\mathbb{R}^N$. In this paper, we study the Bahri–Li minimax method [2] to show that there exist at least three positive solutions of Eq. (1) in Ω.

2. Existence of (PS)-sequences

We define the Palais–Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_h as follows.

Definition 1.

- (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h if $J_h(u_n) = \beta + o(1)$ and $J_h'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;
- (ii) $\beta \in \mathbb{R}$ is a (PS)-value in $H_0^1(\Omega)$ for J_h if there is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h;
- (iii) J_h satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h contains a convergent subsequence.

Lemma 2. *Let $u \in H_0^1(\Omega)$ be a critical point of J_h, then u is a nonnegative solution of Eq.* (1)*. Moreover, if $u \not\equiv 0$ or $h \not\equiv 0$, then u is positive in Ω.*

Proof. Suppose that $u \in H_0^1(\Omega)$ satisfies $\langle J_h'(u), \varphi \rangle = 0$ for any $\varphi \in H_0^1(\Omega)$, that is,

$$\int_\Omega \nabla u \nabla \varphi + u\varphi = \int_\Omega u_+^{p-1}\varphi + h\varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Thus, u is a weak solution of $-\Delta u + u = u_+^{p-1} + h(z)$ in Ω. Since $h \geqslant 0$, by the maximum principle, u is nonnegative. If $u \not\equiv 0$ or $h \not\equiv 0$, we have that u is positive in Ω. $\square$

Let

$$\mathbf{M}_h = \left\{ u \in H_0^1(\Omega)\backslash\{0\} \;\middle|\; u \geqslant 0 \text{ and } \left\langle J_h'(u), u \right\rangle = 0 \right\} \quad \text{and} \quad \alpha_h(\Omega) = \inf_{u \in \mathbf{M}_h} J_h(u).$$

Denote by $\mathbf{M}_0 = \mathbf{M}$, $J_0(u) = J(u)$ and $\alpha_0(\Omega) = \alpha(\Omega)$.

By Chen and Wang [4], we have the following lemmas.

Lemma 3. *There is a bijective $C^{1,1}$ map m from the unit sphere Σ in $H_0^1(\Omega)$ to $\mathbf{M}$. Moreover, $\mathbf{M}$ is path-connected and there exists a constant $c > 0$ such that for any $u \in \mathbf{M}$, $\|u\|_{H^1} \geqslant c$ and $J(u) \geqslant c$.*

Lemma 4.

(i) *For each $u \in H_0^1(\Omega)\backslash\{0\}$, there exists a $s_u > 0$ such that $s_u u \in \mathbf{M}$.*

(ii) *Let $\beta > 0$ and $\{u_n\}$ be a sequence in $H_0^1(\Omega)\backslash\{0\}$ for J such that $J(u_n) = \beta + o(1)$ and $a(u_n) = b(u_n^+) + o(1)$.*

Then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + o(1)$, $\{s_n u_n\}$ in $\mathbf{M}$ and $J(s_n u_n) = \beta + o(1)$.

Lemma 5. *If $u \in H_0^1(\Omega)\backslash\{0\}$, then*

$$\left(\frac{a(u)^{\frac{p}{2}}}{b(u)}\right)^{\frac{1}{p-2}} \geqslant \left(\frac{2p}{p-2}\right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}}.$$

Proof. Applying Lemma 4. $\square$

Lemma 6 *(Palais–Smale Decomposition lemma for J_h). Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h. Then there are a subsequence $\{u_n\}$, a positive integer l, sequences $\{z_n^i\}_{n=1}^\infty$ in $\mathbb{R}^N$, functions u in $H_0^1(\Omega)$, and $w^i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \leqslant i \leqslant l$ such that*

$$\left|z_n^i\right| \to \infty \quad \text{for } 1 \leqslant i \leqslant l;$$

$$-\triangle u + u = |u|^{p-2}u + h(z) \quad \text{in } \Omega; \qquad -\triangle w^i + w^i = \left|w^i\right|^{p-2} w^i \quad \text{in } \mathbb{R}^N;$$

$$u_n = u + \sum_{i=1}^{l} w^i\left(\cdot - z_n^i\right) + o(1) \quad \text{strongly in } H^1\left(\mathbb{R}^N\right);$$

$$J_h(u_n) = J_h(u) + \sum_{i=1}^{l} J\left(w^i\right) + o(1).$$

In addition, if $u_n \geqslant 0$, then $u \geqslant 0$ and $w^i \geqslant 0$ for $1 \leqslant i \leqslant l$.

Proof. See Zhu and Zhou [18]. $\square$

Define $\psi(u) = \langle J_h'(u), u\rangle = a(u) - b(u_+) - \int_\Omega hu$. Then:

Lemma 7. *For each $u \in \mathbf{M}_h$, we have $\langle \psi'(u), u\rangle = a(u) - (p-1)b(u) \neq 0$.*

Proof. By Tarantello [13, Lemma 2.3] and Cao and Zhou [5]. $\square$

By Lemma 7, we write $\mathbf{M}_h = \mathbf{M}_h^+ \cup \mathbf{M}_h^-$, where

$$\mathbf{M}_h^+ = \big\{u \in \mathbf{M}_h \mid a(u) - (p-1)b(u) > 0\big\},$$
$$\mathbf{M}_h^- = \big\{u \in \mathbf{M}_h \mid a(u) - (p-1)b(u) < 0\big\}.$$

Define

$$\alpha_h^+(\Omega) = \inf_{u \in \mathbf{M}_h^+} J_h(u); \qquad \alpha_h^-(\Omega) = \inf_{u \in \mathbf{M}_h^-} J_h(u).$$

By Wang and Wu [16], we have the following lemma.

Lemma 8. *$\{u_n\}$ is a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence in $H_0^1(\Omega)$ for J if and only if $J(u_n) = \alpha(\Omega) + o(1)$ and $a(u_n) = b(u_n^+) + o(1)$. In particular, every minimizing sequence $\{u_n\}$ in $\mathbf{M}$ of $\alpha(\Omega)$ is a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence in $H_0^1(\Omega)$ for J.*

For each nonnegative $u \in H_0^1(\Omega)\backslash\{0\}$, we write

$$t_{\max} = \left(\frac{a(u)}{(p-1)b(u)}\right)^{\frac{1}{p-2}} > 0.$$

Lemma 9. *For each nonnegative $u \in H_0^1(\Omega)\backslash\{0\}$, we have the following results.*

(i) *There is a unique number $t^- = t^-(u) > t_{\max} > 0$ such that $t^- u \in \mathbf{M}_h^-$ and $J_h(t^- u) = \max_{t \geqslant t_{\max}} J_h(tu)$.*
(ii) *$t^-(u)$ is a continuous function.*
(iii) *$\mathbf{M}_h^- = \{u \in H_0^1(\Omega)\backslash\{0\} \mid u \geqslant 0$ and $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1\}$.*
(iv) *If $\int_\Omega hu > 0$, then there is a unique number $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+ u \in \mathbf{M}_h^+$ and $J_h(t^+ u) = \min_{0 \leqslant t \leqslant t^-} J_h(tu)$.*

Proof. See Tarantello [13] and Cao and Zhou [5]. $\square$

Lemma 10.

(i) *For each $u \in \mathbf{M}_h^+$, we have $\int_\Omega hu > 0$ and $J_h(u) < 0$. In particular, $\alpha_h(\Omega) \leqslant \alpha_h^+(\Omega) < 0$.*
(ii) *J_h is coercive and bounded below on $\mathbf{M}_h$.*

Proof. (i) For each $u \in \mathbf{M}_h^+$, $a(u) - (p-1)b(u) > 0$ and $a(u) = b(u) + \int_\Omega hu$. Then

$$\int_\Omega hu = a(u) - b(u) > (p-2)b(u) > 0.$$

Hence

$$J_h(u) = \left(\frac{1}{2} - \frac{1}{p}\right)b(u) - \frac{1}{2}\int_\Omega hu < \frac{p-2}{2p}b(u) - \frac{p-2}{2}b(u) = -\frac{(p-1)(p-2)}{2p}b(u) < 0.$$

(ii) By Tarantello [13, p. 288]. □

Lemma 11. *Let u be in $\mathbf{M}_h$ such that $J_h(u) = \min_{v \in \mathbf{M}_h} J_h(v) = \alpha_h(\Omega)$. Then*

(i) $\int_\Omega hu > 0$;
(ii) *u is a solution of Eq.* (1) *in Ω.*

Proof. (i) By Lemma 10(i), we have

$$0 > \alpha_h(\Omega) = J_h(u) = \left(\frac{1}{2} - \frac{1}{p}\right)a(u) - \left(1 - \frac{1}{p}\right)\int_\Omega hu.$$

Thus, $\int_\Omega hu > 0$.

(ii) By Lemma 7, $\langle \psi'(v), v\rangle \neq 0$ for each $v \in \mathbf{M}_h$. Since $J_h(u) = \min_{v \in \mathbf{M}_h} J_h(v)$, by the Lagrange multiplier theorem, there is a $\lambda \in \mathbb{R}$ such that $J_h'(u) = \lambda\psi'(u)$ in $H^{-1}(\Omega)$. Then we have

$$0 = \left\langle J_h'(u), u\right\rangle = \lambda\left\langle \psi'(u), u\right\rangle.$$

Thus, $\lambda = 0$ and $J_h'(u) = 0$ in $H^{-1}(\Omega)$. Therefore, u is a solution of Eq. (1) in Ω with $J_h(u) = \alpha_h(\Omega)$. □

By Cao and Zhou [5], we have the following two lemmas.

Lemma 12. *Given $u \in \mathbf{M}_h$, then there exist a $\delta > 0$ and a differentiable functional $l : B(0;\delta) \subset H_0^1(\Omega) \to \mathbb{R}^+$ such that $l(0) = 1$, $l(v)(u - v) \in \mathbf{M}_h$ for $v \in B(0;\delta)$ and*

$$\left\langle l'(v), \varphi\right\rangle\Big|_{(l,v)=(1,0)} = \frac{\langle \psi'(u), \varphi\rangle}{\langle \psi'(u), u\rangle} \quad \text{for } \varphi \in C_c^\infty(\Omega).$$

Lemma 13.

(i) *There exists a* $(\mathrm{PS})_{\alpha_h(\Omega)}$*-sequence $\{u_n\}$ in $\mathbf{M}_h$ for J_h.*
(ii) *There exists a* $(\mathrm{PS})_{\alpha_h^+(\Omega)}$*-sequence $\{u_n\}$ in $\mathbf{M}_h^+$ for J_h.*
(iii) *There exists a* $(\mathrm{PS})_{\alpha_h^-(\Omega)}$*-sequence $\{u_n\}$ in $\mathbf{M}_h^-$ for J_h.*

3. Existence of the first solution

By Lemma 13(i), there is a $(\mathrm{PS})_{\alpha_h(\Omega)}$-sequence $\{u_n\}$ in $\mathbf{M}_h$ for J_h. Then we have the following $(\mathrm{PS})_{\alpha_h(\Omega)}$-condition.

Lemma 14. *Let $\{u_n\} \subset \mathbf{M}_h$ be a $(\mathrm{PS})_{\alpha_h(\Omega)}$-sequence for J_h. Then there exist a subsequence $\{u_n\}$ and a nonzero $u_0 \in H_0^1(\Omega)$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$. Moreover, u_0 is a positive solution of Eq.* (1) *such that $J_h(u_0) = \alpha_h(\Omega)$.*

Proof. By Lemma 10(ii), $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Take a subsequence $\{u_n\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Then we have that u_0 is a nonzero solution of Eq. (1) in Ω. Since

$$J_h(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) - \int_\Omega hu_n = \alpha_h(\Omega) + o(1),$$
$$\langle J_h'(u_n), u_n\rangle = a(u_n) - b(u_n) - \int_\Omega hu_n = o(1),$$

we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right)a(u_n) - \left(1 - \frac{1}{p}\right)\int_\Omega hu_n = \alpha_h(\Omega) + o(1).$$

Since the functional a is weakly lower semicontinuous and $\int_\Omega hu_n \to \int_\Omega hu_0$ as $n \to \infty$, then $J_h(u_0) = \alpha_h(\Omega)$. Let $p_n = u_n - u_0$. By the Brézis–Lieb lemma, we get

$$\begin{aligned} J_h(p_n) &= \frac{1}{2}a(p_n) - \frac{1}{p}b(p_n) - \int_\Omega hp_n \\ &= \frac{1}{2}a(u_n) - \frac{1}{2}a(u_0) - \frac{1}{p}b(u_n) + \frac{1}{p}b(u_0) - \int_\Omega hu_n + \int_\Omega hu_0 + o(1) \\ &= J_h(u_n) - J_h(u_0) + o(1) = o(1). \end{aligned} \tag{4}$$

By the Brézis–Lieb lemma, $\int_\Omega hp_n = o(1)$ and u_0 is a solution of Eq. (1), so

$$\begin{aligned} \langle J_h'(p_n), p_n\rangle &= a(p_n) - b(p_n) - \int_\Omega hp_n \\ &= a(u_n) - a(u_0) - b(u_n) + b(u_0) - \int_\Omega hu_n + \int_\Omega hu_0 + o(1) \\ &= \langle J_h'(u_n), u_n\rangle - \langle J_h'(u_0), u_0\rangle = o(1). \end{aligned} \tag{5}$$

Thus, by (4), (5) and $\int_\Omega hp_n = o(1)$, we have

$$\frac{p-2}{2p}a(p_n) = o(1),$$

that is,

$$u_n \to u_0 \quad \text{strongly in } H_0^1(\Omega).$$

Moreover, u_0 is a solution of Eq. (1) such that $J_h(u_0) = \alpha_h(\Omega)$. By Lemma 2, u_0 is positive in Ω. $\square$

We prove that u_0 is the unique critical point of J_h in $B(r_0)$ in the following lemma.

Lemma 15. *Let $r_0 = (\frac{1}{p-1})^{1/(p-2)}(\frac{2p}{p-2})^{1/2}\alpha(\Omega)^{1/2}$. Then*

(i) $\mathbf{M}_h^+ \subset B(r_0) = \{u \in H_0^1(\Omega) \mid \|u\|_{H^1} < r_0\}$;
(ii) *$J_h(u)$ is strictly convex in $B(r_0)$.*

Proof. (i) If $u \in \mathbf{M}_h^+$, then $a(u) > (p-1)b(u)$ and $a(u) = b(u) + \int_\Omega hu$. Thus,

$$a(u) < \frac{1}{p-1}a(u) + \|h\|_{L^2}\|u\|_{H^1}.$$

This implies

$$\begin{aligned}
\|u\|_{H^1} &< \left(\frac{p-1}{p-2}\right)\|h\|_{L^2} < \left(\frac{p-1}{p-2}\right)(p-2)\left(\frac{1}{p-1}\right)^{\frac{p-1}{p-2}}\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\alpha(\Omega)^{\frac{1}{2}} \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\alpha(\Omega)^{\frac{1}{2}} = r_0.
\end{aligned}$$

(ii) We know

$$J_h''(u)(v,v) = a(v) - (p-1)\int_\Omega |u|^{p-2}v^2 \quad \text{for all } v \in H_0^1(\Omega).$$

Thus, by Lemma 5, we obtain

$$\begin{aligned}
&J_h''(u)(v,v) \\
&\quad \geqslant a(v) - (p-1)\|u\|_{L^p}^{p-2}\|v\|_{L^p}^2 \\
&\quad \geqslant a(v) - (p-1)\left[a(u)^{\frac{p-2}{2}}\left(\frac{p-2}{2p}\right)^{\frac{p-2}{2}}\alpha(\Omega)^{-\frac{(p-2)^2}{2p}}\right]\left[a(v)\left(\frac{p-2}{2p}\right)^{\frac{p-2}{p}}\alpha(\Omega)^{\frac{-(p-2)}{p}}\right] \\
&\quad \geqslant a(v)\left[1-(p-1)\left(\frac{2p}{p-2}\alpha(\Omega)\right)^{\frac{2-p}{2}}\|u\|_{H^1}^{p-2}\right] > 0 \quad \text{for } u \in B(r_0)\backslash\{0\}.
\end{aligned}$$

Thus, $J_h''(u)$ is positive definite for $u \in B(r_0)$ and J_h is strictly convex in $B(r_0)$. $\square$

By Lemma 14, there exists a solution $u_0 \in \mathbf{M}_h$ of Eq. (1) such that $J_h(u_0) = \alpha_h(\Omega)$. Furthermore, we have the following lemma.

Lemma 16.

(i) $u_0 \in \mathbf{M}_h^+$ *and* $J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega)$;
(ii) u_0 *is the unique critical point of* $J_h(u)$ *in* $B(r_0)$, *where* r_0 *is defined as in Lemma* 15;
(iii) $J_h(u_0)$ *is a local minimum in* $H_0^1(\Omega)$.

Proof. (i) By Lemma 11(i), $\int_\Omega hu_0 > 0$. We claim that $u_0 \in \mathbf{M}_h^+$. Otherwise, if $u_0 \in \mathbf{M}_h^-$, then by Lemma 9, there exists a unique $t^-(u_0) = 1 > t^+(u_0) > 0$ such that $t^+(u_0)u_0 \in \mathbf{M}_h^+$ and

$$\alpha_h(\Omega) \leqslant \alpha_h^+(\Omega) \leqslant J_h\big(t^+(u_0)u_0\big) < J_h\big(t^-(u_0)u_0\big) = \alpha_h(\Omega),$$

which is a contradiction. Since $u_0 \in \mathbf{M}_h^+$, $\alpha_h^+(\Omega) \leqslant J_h(u_0) = \alpha_h(\Omega) \leqslant \alpha_h^+(\Omega)$, that is, $J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega)$.

(ii) By part (i) and Lemma 15.

(iii) See Cao and Zhou [5, p. 452]. □

Lemma 17. *Let* $u \in H_0^1(\Omega)$ *be a critical point of* J_h, *then either* $u \in \mathbf{M}_h^-$ *or* $u = u_0$.

Proof. Let $u \in H_0^1(\Omega)$ be a critical point of J_h, we get $u \in \mathbf{M}_h = \mathbf{M}_h^+ \cup \mathbf{M}_h^-$. Since $\mathbf{M}_h^+ \cap \mathbf{M}_h^- = \emptyset$, $\mathbf{M}_h^+ \subset B(r_0)$ and u_0 is the unique critical point of $J_h(u)$ in $B(r_0)$, where r_0 is defined as in Lemma 15, then either $u \in \mathbf{M}_h^-$ or $u = u_0$. □

4. Existence of the second solutions

Kwong [8] proved that there is the unique positive solution w of Eq. (3) in $\mathbb{R}^N$ such that $J(w) = \alpha(\mathbb{R}^N)$. Lien, Tzeng and Wang [9] proved that Eq. (3) does not have a positive ground state solution in Ω and $\alpha(\Omega) = \alpha(\mathbb{R}^N)$. Then by Cao and Zhou [5, Proposition 3.1], Palais–Smale Decomposition Lemmas 6 and 17, we have the following restricted $(\mathrm{PS})_\beta$-condition.

Lemma 18.

(i) *If* $\{u_n\}$ *is a* $(\mathrm{PS})_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h *with* $\beta < \alpha_h(\Omega) + \alpha(\Omega)$, *then there exist a subsequence* $\{u_n\}$ *and a nonzero* u^0 *in* $H_0^1(\Omega)$ *such that* $u_n \to u^0$ *strongly in* $H_0^1(\Omega)$ *and* $J_h(u^0) = \beta$. *Moreover,* u^0 *is a positive solution of Eq.* (1) *in* Ω;
(ii) *if* $\{u_n\} \subset \mathbf{M}_h^-$ *is a* $(\mathrm{PS})_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h *with*

$$\alpha_h(\Omega) + \alpha(\Omega) < \beta < \alpha_h^-(\Omega) + \alpha(\Omega),$$

then there exist a subsequence $\{u_n\}$ *and a nonzero* $u^0 \in \mathbf{M}_h^-$ *such that* $u_n \to u^0$ *strongly in* $H_0^1(\Omega)$ *and* $J_h(u^0) = \beta$. *Moreover,* u^0 *is a positive solution of Eq.* (1) *in* Ω.

Proof. (i) Applying the Palais–Smale Decomposition Lemma 6, we get

$$\alpha_h(\Omega) + \alpha(\Omega) > \beta + o(1) = J_h(u_n) = J_h\big(u^0\big) + l\alpha(\Omega) \geqslant \alpha_h(\Omega) + l\alpha(\Omega).$$

Then $l = 0$. Hence, there exist a subsequence $\{u_n\}$ and a nonzero u^0 in $H_0^1(\Omega)$ such that $u_n \to u^0$ strongly in $H_0^1(\Omega)$ and $J_h(u^0) = \beta$. Moreover, u^0 is a positive solution of Eq. (1) in Ω.

(ii) Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there are a subsequence $\{u_n\}$ and a nonzero $u^0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$. By Lemma 17, either $u^0 \in \mathbf{M}_h^-$ or $u^0 = u_0$. Applying the Palais–Smale Decomposition Lemma 6 to obtain

$$\beta + o(1) = J_h(u_n) = J_h\big(u^0\big) + l\alpha(\Omega) \geqslant \alpha_h(\Omega) + l\alpha(\Omega).$$

We know that $\alpha_h^-(\Omega) < \alpha_h(\Omega) + \alpha(\Omega)$, then $l \leqslant 1$. If $l = 1$ and $u^0 = u_0$, then

$$\beta = J_h\big(u^0\big) + \alpha(\Omega) = \alpha_h(\Omega) + \alpha(\Omega),$$

which is a contradiction. If $l = 1$ and $u^0 \in \mathbf{M}_h^-$, then

$$\beta = J_h\big(u^0\big) + \alpha(\Omega) \geqslant \alpha_h^-(\Omega) + \alpha(\Omega),$$

which is a contradiction. Thus, $l = 0$. We complete the proof. $\square$

By Chen, Chen and Wang [3, Proposition 1], we have the following lemma.

Lemma 19. *Let u be a positive solution of Eq.* (1) *in Ω. Then for any $\varepsilon > 0$, there are positive constants c_ε and R such that $D \subset B^N(0; R)$ and*

$$u(z) \geqslant c_\varepsilon \exp\big(-(1+\varepsilon)|z|\big) \quad \textit{for } |z| \geqslant R \textit{ and } z \in \Omega.$$

We know that there is a positive radially symmetric smooth solution w of Eq. (3) in $\mathbb{R}^N$ such that $J(w) = \alpha(\mathbb{R}^N)$. Recall the facts:

(i) for any $\varepsilon > 0$, there exist constants C_0, $C_0' > 0$ such that for all $z \in \mathbb{R}^N$

$$w(z) \leqslant C_0 \exp\big(-|z|\big) \quad \text{and} \quad \big|\nabla w(z)\big| \leqslant C_0' \exp\big(-(1-\varepsilon)|z|\big);$$

(ii) for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$w(z) \geqslant C_\varepsilon \exp\big(-(1+\varepsilon)|z|\big) \quad \text{for all } z \in \mathbb{R}^N.$$

For such R in Lemma 19, let ψ_R be a C^∞-function on $\mathbb{R}^N$ such that $0 \leqslant \psi_R \leqslant 1$, $|\nabla \psi_R| \leqslant c$ and

$$\psi_R(z) = \begin{cases} 1 & \text{for } z_N \geqslant R+1, \\ 0 & \text{for } z_N \leqslant R. \end{cases}$$

We define

$$w_n(z) = \psi_R(z) w(z - n e_N) \quad \text{for } n \in \mathbb{N},$$

where $e_N = (0, \ldots, 0, 1) \in \mathbb{R}^N$. Clearly, $w_n \in H_0^1(\Omega)$.

In order to prove Lemma 24, we need the following lemmas.

Lemma 20.

(i) $a(w_n) = b(w_n) + o(1) = \frac{2p}{p-2}\alpha(\mathbb{R}^N) + o(1)$ *as $n \to \infty$;*

(ii) $J(w_n) = \alpha(\Omega) + o(1) = \alpha(\mathbb{R}^N) + o(1)$ *as* $n \to \infty$;
(iii) $w_n \rightharpoonup 0$ *weakly in* $H_0^1(\Omega)$ *as* $n \to \infty$.

Proof. It is similar to the proof of Wang [15, Lemma 30]. □

Lemma 21. *Let E be a domain in $\mathbb{R}^N$. If $f : E \to \mathbb{R}$ satisfies*

$$\int_E \left| f(z) e^{\sigma |z|} \right| dz < \infty \quad \textit{for some } \sigma > 0,$$

then

$$\left(\int_E f(z) e^{-\sigma |z - ne_N|} \, dz \right) e^{\sigma n} = \int_E f(z) e^{\sigma z_N} \, dz + o(1) \quad \textit{as } n \to \infty.$$

Proof. Since $\sigma |ne_N| \leqslant \sigma |z| + \sigma |z - ne_N|$, we have

$$\left| f(z) e^{-\sigma |z - ne_N|} e^{\sigma |ne_N|} \right| \leqslant \left| f(z) e^{\sigma |z|} \right|.$$

Since $-\sigma |z - ne_N| + \sigma |ne_N| = \sigma \frac{\langle z, ne_N \rangle}{|ne_N|} + o(1)$ as $n \to \infty$, then the lemma follows from the Lebesgue dominated convergence theorem. □

Lemma 22. *For* $t \geqslant 0$, *we have the following inequalities*:

(i) $(1+t)^q \geqslant 1 + t^q + \frac{q}{q-1} t^{q-1}$ *where* $q \geqslant 2$;
(ii) $(1+t)^q \geqslant 1 + t^q + qt$ *where* $q \geqslant 2$;
(iii) $(1+t)^q \geqslant 1 + t^q + qt + \frac{q}{q-2} t^{q-1}$ *where* $q \geqslant 3$;
(iv) *if* $t \leqslant c$ *for some* $c > 0$, *then* $(1+t)^q \geqslant 1 + t^q + qt + A(c)t^2$, *where* $2 < q < 3$ *and* $A(c) > 0$.

Proof. (i) Let $f(t) = (1+t)^q - 1 - t^q - \frac{q}{q-1} t^{q-1}$ for $t \geqslant 0$ and $q \geqslant 2$. Then $f(0) = 0$, and

$$f'(t) = q\left[(1+t)^{q-1} - t^{q-1} - t^{q-2}\right].$$

Since $q \geqslant 2$, we get $(1+t)^{q-1} = (1+t)^{q-2} + t(1+t)^{q-2} \geqslant t^{q-2} + t^{q-1}$. Thus, $f'(t) \geqslant 0$.

(ii) The proof is similar to (i).

(iii) Let $g(t) = (1+t)^q - 1 - t^q - qt - \frac{q}{q-2} t^{q-1}$ for $t \geqslant 0$ and $q \geqslant 3$. Then $g(0) = 0$, and by (i), we obtain

$$g'(t) = q\left[(1+t)^{q-1} - t^{q-1} - 1 - \frac{q-1}{q-2} t^{q-2}\right] \geqslant 0.$$

(iv) Let $h(t) = (1+t)^q - t^q$. Then $h(0) = 1$,

$$h'(t) = q\left[(1+t)^{q-1} - t^{q-1}\right] \quad \text{and} \quad h'(0) = q,$$
$$h''(t) = q(q-1)\left[(1+t)^{q-2} - t^{q-2}\right] > 0.$$

Since $t \leqslant c$ for some $c > 0$, applying the Taylor theorem, we have

$$(1+t)^q - t^q - 1 - qt \geqslant \frac{q(q-1)}{2}\left[(1+c)^{q-2} - c^{q-2}\right]t^2. \qquad \square$$

By Lemma 22, we obtain

$$(a+b)^q \geqslant a^q + b^q + qa^{q-1}b + \frac{q}{q-2}ab^{q-1} \quad \text{for } q \geqslant 3 \text{ and } a, b \geqslant 0, \tag{6}$$

and

$$(a+b)^q \geqslant a^q + b^q + qa^{q-1}b + A(c)a^{q-2}b^2 \quad \text{for } 2 < q < 3 \text{ and } b/a \leqslant c. \tag{7}$$

Lemma 23.

(i) *There exists a number $t_0 > 0$ such that for $0 \leqslant t < t_0$ and each $w_n \in H_0^1(\Omega)$, we have*

$$J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\Omega).$$

(ii) *There exist positive numbers t_1 and n_1 such that for any $t > t_1$ and $n \geqslant n_1$, we have*

$$J_h(tw_n) < 0.$$

Proof. (i) Since J_h is continuous in $H_0^1(\Omega)$ and $\{w_n\}$ is bounded in $H_0^1(\Omega)$, there is a $t_0 > 0$ such that for $0 \leqslant t < t_0$ and each $w_n \in H_0^1(\Omega)$

$$J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\Omega).$$

(ii) By Lemma 20, $J_h(tw_n) = (\frac{t^2}{2} - \frac{t^p}{p})\frac{2p}{p-2}\alpha(\Omega) + o(1)$ as $n \to \infty$. There is an $n_1 > 0$ such that for $n \geqslant n_1$

$$J_h(tw_n) < \left(\frac{t^2}{2} - \frac{t^p}{p}\right)\frac{2p}{p-2}\alpha(\Omega) + 1.$$

Thus, there exists a $t_1 > 0$ such that

$$J_h(tw_n) < 0 \quad \text{for any } t > t_1 \text{ and } n \geqslant n_1. \qquad \square$$

Lemma 24. *There exists a number $n_0 > 0$ such that for $n \geqslant n_0$*

$$\sup_{t \geqslant 0} J_h(u_0 + tw_n) < \alpha_h(\Omega) + \alpha(\Omega),$$

where u_0 is the local minimum in Lemma 16.

Proof. By Lemma 23, we only need to show that there exists an $n_0 > 0$ such that for $n \geqslant n_0$

$$\sup_{t_0 \leqslant t \leqslant t_1} J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\Omega) = \alpha_h(\Omega) + \alpha(\Omega).$$

Since u_0 is a positive solution of Eq. (1) in Ω, then

$$\langle u_0, tw_n\rangle_{H^1} = \int_\Omega \big(u_0^{p-1} tw_n + htw_n\big)\,dz.$$

For $t_0 \leqslant t \leqslant t_1$, since $J(w) = J(w(z - ne_N))$, $\sup_{t\geqslant 0} J(tw) = \alpha(\mathbb{R}^N)$ and $0 \leqslant \psi_R \leqslant 1$, we obtain

$$\begin{aligned}
J_h(u_0 + tw_n) &= \frac{1}{2}\|u_0 + tw_n\|_{H^1}^2 - \frac{1}{p}\int_\Omega (u_0 + tw_n)^p - \int_\Omega h(u_0 + tw_n) \\
&= J_h(u_0) + J(tw_n) + \langle u_0, tw_n\rangle_{H^1} + \frac{1}{p}\int_\Omega u_0^p + (tw_n)^p - (u_0 + tw_n)^p - htw_n \\
&= J_h(u_0) + J(tw_n) - \frac{1}{p}\int_\Omega (u_0 + tw_n)^p - u_0^p - (tw_n)^p - pu_0^{p-1}(tw_n) \\
&\leqslant J_h(u_0) + \alpha\big(\mathbb{R}^N\big) + \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\psi_R|^2\big[w(z - ne_N)\big]^2\,dz \\
&\quad + t^2\int_{\mathbb{R}^N} |\nabla\psi_R|\big|\nabla w(z - ne_N)\big| w(z - ne_N)\,dz \\
&\quad + \frac{t^p}{p}\int_{\mathbb{R}^N} \big(1 - \psi_R^p\big)\big[w(z - ne_N)\big]^p\,dz \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^N} (u_0 + tw_n)^p - u_0^p - (tw_n)^p - pu_0^{p-1}(tw_n).
\end{aligned}$$

For a small $\varepsilon > 0$, since $\operatorname{supp}(1 - \psi_R^p) = \{z \in \mathbb{R}^N \mid z_N \leqslant R + 1\}$ is unbounded, then

$$\int_{\{z_N \leqslant R+1\}} \big(1 - \psi_R^p\big)\big[w(z - ne_N)\big]^p\,dz \leqslant C_1 \exp\big(-(p - \varepsilon)n\big). \tag{8}$$

Similarly, we have

$$\int_{\operatorname{supp}(\nabla\psi_R)} |\nabla\psi_R|^2\big[w(z - ne_N)\big]^2\,dz \leqslant C_2 \exp\big(-(2 - \varepsilon)n\big) \quad \text{and} \tag{9}$$

$$\int_{\operatorname{supp}(\nabla\psi_R)} |\nabla\psi_R|\big|\nabla w(z - ne_N)\big| w(z - ne_N)\,dz \leqslant C_3 \exp\big(-(2 - 2\varepsilon)n\big). \tag{10}$$

(i) For $3 \leqslant p < 2^*$, by (6)

$$\int_{\mathbb{R}^N} (u_0 + tw_n)^p \geqslant \int_{\mathbb{R}^N} u_0^p + (tw_n)^p + pu_0^{p-1}(tw_n) + \frac{p}{p-2} u_0 (tw_n)^{p-1}.$$

Thus, by Lemma 21, for $n \geqslant n_1$

$$\int_{\mathbb{R}^N} u_0 w_n^{p-1}\, dz \geqslant c_1 \exp\bigl(-\min\{1, p-1\}(1+\varepsilon)n\bigr) \geqslant c_1 \exp\bigl(-(1+\varepsilon)n\bigr). \tag{11}$$

Choosing $\varepsilon < 1/3$ and using (8)–(11), we have for $n \geqslant n_1' \geqslant n_1$

$$\sup_{t_0 \leqslant t \leqslant t_1} J_h(u_0 + tw_n) < J_h(u_0) + \alpha\bigl(\mathbb{R}^N\bigr).$$

(ii) For $2 < p < 3$, by Lemma 22(ii), we get

$$(I) = (u_0 + tw_n)^p - u_0^p - (tw_n)^p - pu_0^{p-1}(tw_n) \geqslant 0.$$

Then

$$\int_{\mathbb{R}^N} (I)\, dz \geqslant \int_{\{|z| \leqslant 2R\}} (I)\, dz. \tag{12}$$

Since $\max\{w_n(z)/u_0(z) \mid |z| \leqslant 2R\} \leqslant c < \infty$ for each $n \in \mathbb{N}$, by (7)

$$\int_{\{|z| \leqslant 2R\}} (u_0 + tw_n)^p \geqslant \int_{\{|z| \leqslant 2R\}} u_0^p + (tw_n)^p + pu_0^{p-1}(tw_n) + A(c) u_0^{p-2}(tw_n)^2.$$

Thus, by Lemma 21, for $n \geqslant n_2$

$$\int_{\{|z| \leqslant 2R\}} u_0^{p-2} w_n^2\, dz \geqslant c_2 \exp\bigl(-\min\{2, p-2\}(1+\varepsilon)n\bigr) \geqslant c_2 \exp\bigl(-(p-2)(1+\varepsilon)n\bigr). \tag{13}$$

Choosing $\varepsilon < (4-p)/p$ and using (8)–(10), (12), (13), we have for $n \geqslant n_2' \geqslant n_2$

$$\sup_{t_0 \leqslant t \leqslant t_1} J_h(u_0 + tw_n) < J_h(u_0) + \alpha\bigl(\mathbb{R}^N\bigr) = \alpha_h(\Omega) + \alpha(\Omega).$$

Let $n_0 = \max\{n_1', n_2'\}$, we complete the proof. $\square$

Let

$$A_1 = \left\{ u \in H_0^1(\Omega)\backslash\{0\} \;\middle|\; u \geqslant 0 \text{ and } \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) > 1 \right\} \cup \{0\},$$

$$A_2 = \left\{ u \in H_0^1(\Omega)\backslash\{0\} \;\middle|\; u \geqslant 0 \text{ and } \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) < 1 \right\}.$$

From Tarantello [13], we have the following results.

Lemma 25.

(i) $A\backslash \mathbf{M}_h^- = A_1 \cup A_2$, *where* $A = \{u \in H_0^1(\Omega) \mid u \geqslant 0\}$;
(ii) $\mathbf{M}_h^+ \subset A_1$;
(iii) *there exist* $t_0 > 1$ *and* $n_1 \geqslant n_0$ *such that* $u_0 + t_0 w_n \in A_2$ *for each* $n \geqslant n_1$, *where* n_0 *is defined as in Lemma* 24;
(iv) *there exists a sequence* $\{s_n\} \subset (0, 1)$ *such that* $u_0 + s_n t_0 w_n \in \mathbf{M}_h^-$ *for each* $n \geqslant n_1$;
(v) $\alpha_h^- < \alpha_h(\Omega) + \alpha(\Omega)$.

Proof. (i) By Lemma 9(iii).

(ii) For each $u \in \mathbf{M}_h^+$, we have

$$1 < t_{\max}(u) < t^-(u) = \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right),$$

then $\mathbf{M}_h^+ \subset A_1$. In particular, $u_0 \in A_1$, where u_0 is defined as in Lemma 16.

(iii) There is a constant $c > 0$ such that $0 < t^-(\frac{u_0+tw_n}{\|u_0+tw_n\|_{H^1}}) < c$ for each $t \geqslant 0$ and each $n \in \mathbb{N}$. Otherwise, there exist a sequence $\{t_n\}$ and a subsequence $\{w_n\}$ such that $t^-(\frac{u_0+t_nw_n}{\|u_0+t_nw_n\|_{H^1}}) \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_0+t_nw_n}{\|u_0+t_nw_n\|_{H^1}}$. We claim that $b(v_n)$ is bounded below away from zero.

Case (a): there is a subsequence $\{t_n\}$ such that $t_n = c_0 + o(1)$ as $n \to \infty$, where $c_0 > 0$. By Lemma 20, we have

$$a(w_n) = b(w_n) + o(1) = \frac{2p}{p-2}\alpha(\Omega) + o(1).$$

Thus,

$$\begin{aligned} b(v_n) &= \frac{1}{\|\frac{u_0}{t_n} + w_n\|_{H^1}^p} \int_\Omega \left(\frac{u_0}{t_n} + w_n\right)^p \geqslant \frac{b(w_n)}{2^{p-1}\left(\|\frac{u_0}{t_n}\|_{H^1}^p + \|w_n\|_{H^1}^p\right)} \\ &= \frac{\frac{2p}{p-2}\alpha(\Omega)}{2^{p-1}(\|u_0\|_{H^1}^p / c_0^p + ((2p/(p-2))\alpha(\Omega))^{\frac{p}{2}})} + o(1). \end{aligned}$$

Case (b): $t_n \to \infty$ as $n \to \infty$. The proof is similar to case (a).

Case (c): there is a subsequence $\{t_n\}$ such that $t_n = o(1)$ as $n \to \infty$. By Lemma 20, we have

$$\|u_0 + t_n w_n\|_{H^1}^2 = \|u_0\|_{H^1}^2 + t_n^2\|w_n\|_{H^1}^2 + 2t_n\langle w_n, u_0\rangle_{H^1} = \|u_0\|_{H^1}^2 + o(1).$$

Thus,

$$b(v_n) \geqslant \frac{1}{\|u_0 + t_n w_n\|_{H^1}^p} \int_\Omega u_0^p = \frac{1}{\|u_0\|_{H^1}^p} \int_\Omega u_0^p + o(1).$$

Since $t^-(v_n)v_n \in \mathbf{M}_h^- \subset \mathbf{M}_h$, we have

$$J_h\big(t^-(v_n)v_n\big) = \frac{1}{2}\big[t^-(v_n)\big]^2 - \frac{1}{p}\big[t^-(v_n)\big]^p b(v_n) - t^-(v_n)\int_\Omega hv_n \to -\infty \quad \text{as } n \to \infty.$$

However, J_h is bounded below on $\mathbf{M}_h$, which is a contradiction. Let

$$t_0 = \left(\frac{p-2}{2p\alpha(\Omega)}\left|c^2 - a(u_0)\right|\right)^{\frac{1}{2}} + 1,$$

then

$$\begin{aligned}\|u_0 + t_0 w_n\|_{H^1}^2 &= a(u_0) + t_0^2\left(\frac{2p}{p-2}\right)\alpha(\Omega) + o(1) \\ &> c^2 + o(1) \geqslant \left[t^-\left(\frac{u_0 + t_0 w_n}{\|u_0 + t_0 w_n\|_{H^1}}\right)\right]^2 + o(1).\end{aligned}$$

Thus, there is an $n_1 \geqslant n_0$, where n_0 is defined as in Lemma 24, such that, for $n \geqslant n_1$,

$$\frac{1}{\|u_0 + t_0 w_n\|_{H^1}} t^-\left(\frac{u_0 + t_0 w_n}{\|u_0 + t_0 w_n\|_{H^1}}\right) < 1,$$

or $u_0 + t_0 w_n \in A_2$.

(iv) Define a path $\gamma_n(s) = u_0 + st_0 w_n$ for $s \in [0, 1]$ and each $n \geqslant n_1$, where $t_0 > 1$, then

$$\gamma_n(0) = u_0 \in A_1, \qquad \gamma_n(1) = u_0 + t_0 w_n \in A_2.$$

Since $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$ is a continuous function for nonzero u and $\gamma_n([0, 1])$ is connected, there exists a sequence $\{s_n\} \subset (0, 1)$ such that $u_0 + s_n t_0 w_n \in \mathbf{M}_h^-$.

(v) By part (iv) and Lemma 24,

$$\alpha_h^- \leqslant J_h(u_0 + s_n t_0 w_n) < J_h(u_0) + \alpha(\Omega) = \alpha_h(\Omega) + \alpha(\Omega). \qquad \square$$

For $c > 0$, we define

$$b_c(u) = \int_\Omega cu^p, \qquad I_c(u) = \frac{1}{2}a(u) - \frac{1}{p}b_c(u_+),$$

$$\mathbf{M}_{I_c} = \big\{u \in H_0^1(\Omega)\backslash\{0\} \;\big|\; \big\langle I_c'(u), u\big\rangle = 0\big\}.$$

Recall that there exist a unique $t^- = t^-(u) > 0$ and a unique $t^1 = t^1(u) > 0$ such that $t^- u \in \mathbf{M}_h^-$ and $t^1 u \in \mathbf{M}$. Let $\Sigma = \{u \in H_0^1(\Omega) \mid u \geqslant 0 \text{ and } \|u\|_{H^1} = 1\}$. Then we have the following results.

Lemma 26.

(i) *For each $u \in \Sigma$, there exists a unique number $t^c(u) > 0$ such that $t^c(u)u \in \mathbf{M}_{I_c}$ and*

$$\max_{t\geqslant 0} I_c(tu) = I_c\big(t^c(u)u\big) = \left(\frac{1}{2} - \frac{1}{p}\right) b_c(u)^{-\frac{2}{p-2}}.$$

(ii) *For each nonnegative $u \in H_0^1(\Omega)$ and $0 < \mu < 1$, we have*

$$(1-\mu) I_{1/(1-\mu)}(u) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leqslant J_h(u) \leqslant (1+\mu) I_{1/(1+\mu)}(u) + \frac{1}{2\mu}\|h\|_{L^2}^2.$$

(iii) *For each $u \in \Sigma$ and $0 < \mu < 1$, we have*

$$(1-\mu)^{\frac{p}{p-2}} J(t^1 u) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leqslant J_h(t^- u) \leqslant (1+\mu)^{\frac{p}{p-2}} J\big(t^1 u\big) + \frac{1}{2\mu}\|h\|_{L^2}^2.$$

(iv) $\alpha_h^- > 0$ *for sufficiently small $\|h\|_{L^2}$.*

Proof. (i) For each $u \in \Sigma$, let $f(t) = I_c(tu) = \frac{1}{2}t^2 - \frac{1}{p}t^p b_c(u)$, then $f(t) \to -\infty$ as $t \to \infty$, $f'(t) = t - t^{p-1} b_c(u)$ and $f''(t) = 1 - (p-1)t^{p-2} b_c(u)$. Let

$$t^c(u) = \left(\frac{1}{b_c(u)}\right)^{\frac{1}{p-2}} > 0.$$

Then $f'(t^c(u)) = 0$, $t^c(u)u \in \mathbf{M}_{I_c}$ and

$$\big(t^c(u)\big)^2 f''\big(t^c(u)\big) = a\big(t^c(u)u\big) - (p-1) b_c\big(t^c(u)u\big) = (2-p)\big(t^c(u)\big)^2 a(u) < 0.$$

Thus, there exists a unique $t^c(u) > 0$ such that $t^c(u)u \in \mathbf{M}_{I_c}$ and

$$\max_{t\geqslant 0} I_c(tu) = I_c\big(t^c(u)u\big) = \left(\frac{1}{2} - \frac{1}{p}\right) b_c(u)^{-\frac{2}{p-2}}.$$

(ii) For $\mu \in (0,1)$, we get

$$\left|\int_\Omega hu\,dz\right| \leqslant \|u\|_{H^1}\|h\|_{L^2} \leqslant \frac{\mu}{2}\|u\|_{H^1}^2 + \frac{1}{2\mu}\|h\|_{L^2}^2.$$

Thus, for each nonnegative $u \in H_0^1(\Omega)$, then

$$\frac{1-\mu}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_\Omega u^p - \frac{1}{2\mu}\|h\|_{L^2}^2 \leqslant J_h(u) \leqslant \frac{1+\mu}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_\Omega u^p + \frac{1}{2\mu}\|h\|_{L^2}^2.$$

(iii) Applying part (ii), we have that for each $u \in \Sigma$

$$(1-\mu)I_{1/(1-\mu)}\big(t^{c_1}u\big) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leqslant J_h(t^- u) \leqslant (1+\mu)I_{1/(1+\mu)}\big(t^{c_2}u\big) + \frac{1}{2\mu}\|h\|_{L^2}^2,$$

where $t^{c_1}u \in \mathbf{M}_{I_{1/(1-\mu)}}$ and $t^{c_2}u \in \mathbf{M}_{I_{1/(1+\mu)}}$. By part (i), then

$$\begin{aligned} I_{1/(1-\mu)}\big(t^{c_1}u\big) &= \left(\frac{1}{2}-\frac{1}{p}\right) b_{1/(1-\mu)}(u)^{-\frac{2}{p-2}} = (1-\mu)^{\frac{2}{p-2}}\left(\frac{1}{2}-\frac{1}{p}\right) b(u)^{-\frac{2}{p-2}} \\ &= (1-\mu)^{\frac{2}{p-2}} J\big(t^1 u\big). \end{aligned}$$

Similarly, $I_{1/(1+\mu)}(t^{c_2}u) = (1+\mu)^{\frac{2}{p-2}} J(t^1 u)$. Hence, (iii) holds.

(iv) We know that $\alpha(\Omega) > 0$ and applying part (iii) to obtain

$$\alpha_h^- \geqslant (1-\mu)^{\frac{p}{p-2}}\alpha(\Omega) - \frac{1}{2\mu}\|h\|_{L^2}^2.$$

Thus, (iv) holds. □

By Lemma 13(iii), there is a $(\mathrm{PS})_{\alpha_h^-(\Omega)}$-sequence $\{u_n\}$ in $\mathbf{M}_h^-$ for J_h. Then we have the following $(\mathrm{PS})_{\alpha_h^-(\Omega)}$-condition.

Lemma 27. *Let $\{u_n\} \subset \mathbf{M}_h^-$ be a $(\mathrm{PS})_{\alpha_h^-(\Omega)}$-sequence for J_h. Then there exist a subsequence $\{u_n\}$ and a nonzero $u^0 \in H_0^1(\Omega)$ such that $u_n \to u^0$ strongly in $H_0^1(\Omega)$. Moreover, u^0 is a positive solution of Eq.* (1) *such that $J_h(u^0) = \alpha_h^-(\Omega)$.*

Proof. By Lemma 25(v), $\alpha_h^-(\Omega) < \alpha_h(\Omega) + \alpha(\Omega)$. Then applying Lemma 18(i), we have that there exists a positive solution u^0 of Eq. (1) such that $J_h(u^0) = \alpha_h^-(\Omega)$. □

Therefore, by Lemmas 2, 14 and 27, Eq. (1) admits at least two positive solutions in Ω.

Theorem 28. *Assume that $h(z) \geqslant 0$ and $0 < \|h\|_{L^2} < d(p,\alpha)$, then there are at least two positive solutions of Eq.* (1) *in Ω.*

Remark 1. After the simple modification, we also prove that Eq. (1) admits at least two positive solutions in a large domain Ω in $\mathbb{R}^N$, that is, for any $r > 0$, there exists a $z \in \Omega$ such that $B^N(z;r) \subset \Omega$.

5. Existence of the third solution

Since $\alpha_h^- > 0$ for sufficiently small $\|h\|_{L^2}$, we define

$$K_h(u) = \sup_{t \geqslant 0} J_h(tu) = J_h(t^- u) > 0,$$

where $t^- u \in \mathbf{M}_h^-$. We observe that if $\|h\|_{L^2}$ is sufficiently small, the Bahri–Li minimax argument [2] also works for K_h. Let

$$\Gamma = \left\{ g \in C\left(\overline{B_r(0)}, \Sigma\right) | g|_{\partial B_r(0)} = \psi_R(z) w(z-y) / \left\| \psi_R(z) w(z-y) \right\|_{H^1} \right\} \quad \text{for large } r = |y|,$$

where $\Sigma = \{u \in H_0^1(\Omega) \mid u \geqslant 0 \text{ and } \|u\|_{H^1} = 1\}$. Then we define

$$\gamma_h(\Omega) = \inf_{g \in \Gamma} \sup_{y \in \mathbb{R}^N} K_h\big(g(y)\big), \qquad \gamma_0(\Omega) = \inf_{g \in \Gamma} \sup_{y \in \mathbb{R}^N} K_0\big(g(y)\big).$$

By Lemma 26(iii), for $0 < \mu < 1$, we have

$$(1-\mu)^{\frac{p}{p-2}} \gamma_0(\Omega) - \frac{1}{2\mu} \|h\|_{L^2}^2 \leqslant \gamma_h(\Omega) \leqslant (1+\mu)^{\frac{p}{p-2}} \gamma_0(\Omega) + \frac{1}{2\mu} \|h\|_{L^2}^2. \tag{14}$$

We know that $\Omega = \mathbb{R}_+^N \backslash \overline{D}$ is the half space with a hole. Throughout this section, assume that D is small and far away from the axis z', that is, $\operatorname{diam}(D)$ and $\operatorname{dist}(D, (z', 0))$ are sufficiently small. Then we have the following important lemma.

Lemma 29. $\alpha(\Omega) < \gamma_0(\Omega) < 2\alpha(\Omega)$.

Proof. Wang [14] proved that Eq. (3) admits at least one positive solution u in Ω and $J(u) = \gamma_0(\Omega) < 2\alpha(\Omega)$. Lien, Tzeng and Wang [9] proved that Eq. (3) does not have a positive ground state solution in Ω and $\alpha(\Omega) = \alpha(\mathbb{R}^N)$. Hence, $\alpha(\Omega) < \gamma_0(\Omega) < 2\alpha(\Omega)$. □

The following minimax theorem is given in Shi [12] to unify the mountain pass lemma of Ambrosetti and Rabinowitz [1] and the saddle point theorem of Rabinowitz [10].

Theorem 30. *Let K be a compact metric space, $K_0 \subset K$ a closed set, X a Banach space, $\chi \in C(K_0, X)$ and let us define the complete metric space M by*

$$M = \left\{ g \in C(K, X) \mid g(s) = \chi(s) \text{ if } s \in K_0 \right\}$$

with the usual distance d. Let $\varphi \in C^1(X, \mathbb{R})$ and let us define

$$c = \inf_{g \in M} \max_{s \in K} \varphi\big(g(s)\big), \qquad c_1 = \max_{\chi(K_0)} \varphi.$$

If $c > c_1$, then for each $\varepsilon > 0$ and each $f \in M$ such that

$$\max_{s \in K} \varphi\big(f(s)\big) \leqslant c + \varepsilon,$$

there exists $v \in X$ such that

$$c - \varepsilon \leqslant \varphi(v) \leqslant \max_{s \in K} \varphi\big(f(s)\big), \qquad \operatorname{dist}\big(v, f(K)\big) \leqslant \varepsilon^{1/2}, \qquad \left\|\varphi'(v)\right\| \leqslant \varepsilon^{1/2}.$$

Lemma 31. *There exists a number $d_0 > 0$ such that if $0 < \|h\|_{L^2} < d_0$, then*

$$\alpha_h(\Omega) + \alpha(\Omega) < \gamma_h(\Omega) < \alpha_h^-(\Omega) + \alpha(\Omega).$$

Moreover, there exists a positive solution u of Eq. (1) *in Ω such that $J_h(u) = \gamma_h(\Omega)$.*

Proof. By Lemma 26(iii), we also have that for $0 < \mu < 1$

$$(1-\mu)^{\frac{p}{p-2}}\alpha(\Omega) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leqslant \alpha_h^-(\Omega) \leqslant (1+\mu)^{\frac{p}{p-2}}\alpha(\Omega) + \frac{1}{2\mu}\|h\|_{L^2}^2.$$

For any $\varepsilon > 0$, there exists a $d_1(\varepsilon) > 0$ such that if $\|h\|_{L^2} < d_1(\varepsilon)$, then

$$\alpha(\Omega) - \varepsilon < \alpha_h^-(\Omega) < \alpha(\Omega) + \varepsilon.$$

Thus,

$$2\alpha(\Omega) - \varepsilon < \alpha_h^-(\Omega) + \alpha(\Omega) < 2\alpha(\Omega) + \varepsilon.$$

Using (14), for any $\delta > 0$, there exists a $d_2(\delta) > 0$ such that if $\|h\|_{L^2} < d_2(\delta)$, then

$$\gamma_0(\Omega) - \delta < \gamma_h(\Omega) < \gamma_0(\Omega) + \delta.$$

Fix a small $0 < \varepsilon < (2\alpha(\Omega) - \gamma_0(\Omega))/2$, since $\alpha(\Omega) < \gamma_0(\Omega) < 2\alpha(\Omega)$, choosing a $\delta > 0$ such that for $\|h\|_{L^2} < d_0 = \min\{d_1, d_2\}$, we get

$$\alpha_h(\Omega) + \alpha(\Omega) < \alpha(\Omega) < \gamma_h(\Omega) < 2\alpha(\Omega) - \varepsilon < \alpha_h^-(\Omega) + \alpha(\Omega).$$

Since

$$\begin{aligned} K_h\big(\psi_R(z)w(z-y)/\big\|\psi_R(z)w(z-y)\big\|_{H^1}\big) &= J_h\big(t^-\psi_R(z)w(z-y)/\big\|\psi_R(z)w(z-y)\big\|_{H^1}\big) \\ &= \alpha\big(\mathbb{R}^N\big) + o(1) = \alpha(\Omega) + o(1) \quad \text{as } |y| \to \infty, \end{aligned}$$

then $\gamma_h(\Omega) > K_h(\psi_R(z)w(z-y)/\|\psi_R(z)w(z-y)\|_{H^1})$ for large $r = |y|$. Applying the minimax Theorem 30 to obtain that $\gamma_h(\Omega)$ is a (PS)-value in $H_0^1(\Omega)$ for J_h. Therefore, by Lemmas 2 and 18(ii), we have that there exists a positive solution u of Eq. (1) in Ω such that $J_h(u) = \gamma_h(\Omega)$. □

We can conclude the following theorem.

Theorem 32. *Assume that $h(z) \geqslant 0$ and $0 < \|h\|_{L^2} < \min\{d(p,\alpha), d_0\}$, where d_0 is defined as in Lemma* 31. *If* $\operatorname{diam}(D)$ *and* $\operatorname{dist}(D, (z', 0))$ *are sufficiently small, then there are at least three positive solutions of Eq.* (1) *in Ω.*

Proof. By Lemmas 2, 14, 27 and 31, we have that Eq. (1) has at least three positive solutions in Ω. □

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